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Dynkin diagrams of hyperbolic Kac–Moody superalgebras

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Abstract
Hyperbolic Kac–Moody superalgebras are classified in terms of their Dynkin diagrams. These types of Kac–Moody superalgebras are those whose diagrams revert to either that of simple or affine superalgebras upon deletion of one of the vertices. It is found that the maximum rank of this type of algebra is 6.

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1. Introduction

Recently, we have seen that hyperbolic Kac–Moody algebras [1–4] could account for a variety of problems in the realms of string theory [5] (hyperbolic Kac–Moody algebra E_{10}), duality properties of supersymmetric gauge theories [6] and two-dimensional field theories [7]. So it is natural to visualize, evaluate and interpret the possible consequences of supersymmetric extension of these algebras. The constructs so generated are the hyperbolic Kac–Moody superalgebras, which can be readily transcribed for the purpose of application to both bosonic and fermionic sectors in a systematic and consistent framework. Our aim in this paper is to take a simple step towards the characterization of such algebras by enumerating their Dynkin diagrams, quite similar to the characterization of hyperbolic Kac–Moody algebra as done by Saclioglu [8].

The procedure for constructing Dynkin diagrams of such algebras is quite close to that of hyperbolic Kac–Moody algebras. But in contrast to the case of hyperbolic Kac–Moody algebra, where there is only one simple root system for each individual algebra, the superalgebraic structure is endowed with several inequivalent root systems, due to the presence of both even and odd roots. This leads to a non-trivial amount of work and a surprising proliferation of cases (we properly acknowledge Saclioglu for the use of these words). However, in this paper we have given the Dynkin diagrams of such algebras in distinguished bases. A distinguished basis means there is the presence of a single odd root in the simple root system of the corresponding simple Lie superalgebra and at most two odd roots in the simple root systems of the corresponding affine Kac–Moody superalgebras from which the hyperbolic superalgebras are constructed. It is interesting to note that the maximum allowed rank of hyperbolic Kac–Moody superalgebra is 6 and there is no such algebra of rank higher than 6. Furthermore, although there are infinitely many hyperbolic Kac–Moody superalgebras of rank 2, the number of these algebras of rank from 3 to 6 is necessarily finite.

This paper is organized as follows. In section 2, we briefly define the hyperbolic and strictly hyperbolic Dynkin diagrams and give a hint for constructing the Dynkin diagrams of hyperbolic Kac–Moody superalgebras. In section 3, we give all the Dynkin diagrams of hyperbolic Kac–Moody superalgebras in distinguished bases and we also show how these Dynkin diagrams can be used to determine the regular sub-superalgebras of a given algebra. Section 4 contains a few concluding remarks.

2. Kac–Moody superalgebra and Dynkin diagrams

A Kac–Moody superalgebra G of rank r can be characterized by a Cartan matrix a_{ij} and a subset $\tau \subset \mathrm{I} \equiv \{1, 2, \ldots, r\}$ that identifies the odd generators. Unless G is an ordinary Lie algebra, in which case $\tau = \varphi$, the set τ can actually be taken to consist of only one element in distinguished basis. Let [,] stand for the graded product defined by $[x, y] = -(-1)^{\deg x \deg y}[y, x]$ and $[x, [y, z]] = [[x, y], z] + (-1)^{\deg x \deg y}[y, [z, x]]$ and we denote as usual by (ad x) the adjoint operation (ad x) $y = [x, y]$. The algebra G can be constructed from the $3r$ generators, $\hat{e}_i$, $\hat{f}_i$ and $\hat{h}_i$, $i \in \mathrm{I}$ which satisfy the relations,

$$
\begin{array}{lll}
[\hat{e}_i, \hat{f}_i] = \delta_{ij}\hat{h}_i & [\hat{h}_i, \hat{h}_j] = 0 & \\
[\hat{h}_i, \hat{e}_j] = a_{ij}\hat{e}_j & [\hat{h}_i, \hat{f}_j] = -a_{ij}\hat{f}_j & \\
(\mathrm{ad}\,\hat{e}_i)^{1-\hat{a}_{ij}}\hat{e}_j = 0 & (\mathrm{ad}\,\hat{f}_i)^{1-\hat{a}_{ij}}\hat{f}_j = 0 & i \neq j
\end{array}
\tag{2.1}
$$

with

$$
\begin{array}{lll}
\deg \hat{h}_i = 0 & \deg \hat{e}_i = \deg \hat{f}_i = 0 & i \notin \tau \\
 & \deg \hat{e}_i = \deg \hat{f}_i = 1 & i \in \tau
\end{array}
$$

$\hat{a}_{ij}$ is the matrix which is obtained from the non-symmetric Cartan matrix a_{ij} by substituting –1 for the strictly positive elements in the rows with '0' on the diagonal entry. In the case of Lie algebras the matrices a_{ij} and $\hat{a}_{ij}$ coincide and equation (2.1) reduces to the standard Serre relations. However, in the case of superalgebra the description given by the above Serre relation leads in general to a larger superalgebra than the superalgebra under consideration. So it is necessary to write supplementary relations involving more than two generators, in order to quotient the larger superalgebra and to recover the original one. These supplementary conditions appear when we deal with odd roots of zero length (i.e. $\alpha_{ii} = 0$). The supplementary conditions depend on the different types of vertices which appear in the Dynkin diagrams. For example, in the case of A (m, n), if α_i is an odd root then the supplementary condition,

$$
[[[\hat{e}_{i-1}, \hat{e}_i], \hat{e}_{i+1}], \hat{e}_i] = 0 \tag{2.1a}
$$

is necessary. Similarly different types of relations hold good for different types of vertices, the details of which can be found in [13, 15].

In equation (2.1) the matrix α_{ij} is symmetrizable and indecomposable. An indecomposable Cartan matrix A is that which cannot be reduced to a block diagonal form and a symmetrizable GCM is that which can be expressed as $A = DG$, where D is a diagonal matrix and G is a symmetric matrix with entries of D and G are rational numbers in general.

Now, we can think of the symmetric matrix G_{ij} as a metric on a root space and we make the following identification,

$$\begin{aligned} D_{ij} &= \frac{2}{(\alpha_i, \alpha_i)}\delta_{ij} && \text{where } \alpha_i \text{ is an even simple root, i.e. } i \notin \tau\text{, or a non-degenerate odd root, i.e. } i \in \tau\text{, but } 2\alpha_i \text{ is also a root} \\ &= \delta_{ij} && \text{where } \alpha_i \text{ is a degenerate odd root, i.e. } i \in \tau\text{, and } (\alpha_i, \alpha_i) = 0 \end{aligned}$$

and

$$G_{ij} = (\alpha_i, \alpha_j). \tag{2.2}$$

To each simple root system of the algebras a Dynkin diagram can be associated according to the following rules:

(i) To each simple bosonic root we associate a white dot '○', to each simple fermionic one α_i a block dot '●' if $a_{ii} \neq 0$ (i.e. $2\alpha_i \in \Delta_0$) and a grey dot '⊗' if $a_{ii} = 0$.
The ith and jth dots will be joined by η_{ij} lines with

$$\begin{aligned} \eta_{ij} &= \frac{2|a_{ij}|}{\min(|a_{ij}|, |a_{ij}|)} && \text{if} \quad a_{ii}.\, a_{jj} \neq 0 \\ \eta_{ij} &= \frac{2|a_{ij}|}{\min a_{kk} \neq 0 |a_{kk}|} && \text{if} \quad a_{ii} \neq 0 \quad a_{jj} = 0 \\ \eta_{ij} &= |a_{ij}| && \text{if} \quad a_{ii} = a_{jj} = 0. \end{aligned} \tag{2.3}$$

(ii) We add an arrow on the lines connecting the ith and jth dots when $\eta_{ij} > 1$ pointing from i and j if $a_{ii}.\, a_{jj} \neq 0$ and $|a_{jj}|$ or if $a_{ii} = 0, a_{jj} \neq 0, |a_{jj}| < 2$ and pointing from j to i if $a_{ii} = 0, a_{jj} \neq 0, |a_{jj}| > 2$.

Given the class of symmetrizable GCM and their associated algebras we consider three types of superalgebras: simple Lie superalgebras, affine Kac–Moody algebras and hyperbolic Kac–Moody superalgebras. The first two types of superalgebras have already been classified and studied in detail. For the sake of completeness we first give the list of different families of finite-dimensional simple Lie superalgebras:

(I) $A(m, n)$ or $sl\,(m + 1, n + 1)$
(II) $B(m, n)$ or $osp\,(2m + 1, 2n)$ with $m \neq 0$
(III) $B(0, n)$ or $osp\,(1, 2n)$
(IV) $D(m, n)$ or $osp\,(2m, 2n)$ with $m \neq 1$
(V) $D(2, 1; \alpha)$
(VI) $F(4)$
(VII) $G(3)$.

Similarly, the affine Kac–Moody superalgebras are given by

(i) $sl^{(1)}\,(m, n)$
(ii) $osp^{(1)}\,(2m + 1, 2n)$
(iii) $osp^{(1)}\,(1, 2n)$
(iv) $osp^{(1)}\,(2m, 2n)$
(v) $D^{(1)}\,(2, 1; \alpha)$
(vi) $F^{(1)}(4)$
(vii) $G^{(1)}\,(3)$
(viii) $sl^{(2)}\,(2m, 2n)$
(ix) $sl^{(2)}\,(2m + 1, 2n)$

(x) $sl^{(2)}(2m+1, 2n+1)$
(xi) $osp^{(2)}(2m, 2n)$
(xii) $sl^{(4)}(2m+1, 2n-2m+1)$ $(n \geqslant 1, 0 \leqslant m \leqslant n, 2m \neq n)$
(xiii) $A^{(4)}(n, n)$ $(n \geqslant 2, n = \text{even})$.

The Dynkin diagrams of simple Lie superalgebras and affine Kac–Moody superalgebras have been discussed in detail by various authors such as Kac [2], Yamane [13], Frappat, Sciarrino and Sorba [4] and Van De Leur [14].

In this paper, we concentrate only on the classification of hyperbolic Kac–Moody superalgebra through Dynkin diagrams. It consists of two sub-classes: hyperbolic Dynkin diagrams, which revert to the Dynkin diagrams of simple Lie superalgebras or affine Kac–Moody superalgebras upon the deletion of any vertex of the diagram, and strictly hyperbolic Dynkin diagrams, which yield only simple Lie superalgebra Dynkin diagrams under the same operation. The general strategy in constructing the hyperbolic Dynkin diagrams of rank $(r+1)$ is as follows. First we draw all possible Lie and/or affine diagrams of rank r and then add an extra root trying all possible lengths. Then we try connecting the new root to the old ones in all possible ways consistent with a symmetrizable Cartan matrix. Finally, we test the resulting diagram by removing any vertex to see whether it reduces to simple or affine superalgebra (semi-simple algebras are also allowed). A diagram that survives the above operation is of hyperbolic type. In the next section, we draw all possible Dynkin diagrams of hyperbolic Kac–Moody superalgebras in the distinguished bases. One practical way to obtain all the simple root systems (distinguished and non-distinguished) or equivalently all the Dynkin diagrams of the given Kac–Moody superalgebra is to apply a set of transformations to a given Dynkin diagram [4]. For example, the Weyl reflection relative to the even roots is given by

$$\sigma_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha \tag{2.4}$$

where $\alpha \in \Delta_0$ and $\beta \in \Delta_0 \cup \Delta_1$ with Δ_0 and Δ_1 being sets of even and odd roots of the algebra, respectively. When this transformation is applied to a simple root system then an equivalent root system is obtained with the same Dynkin diagram. Similarly, now we can consider the set of transformations associated to the odd roots, i.e. $\alpha \in \Delta_1$, which are given as

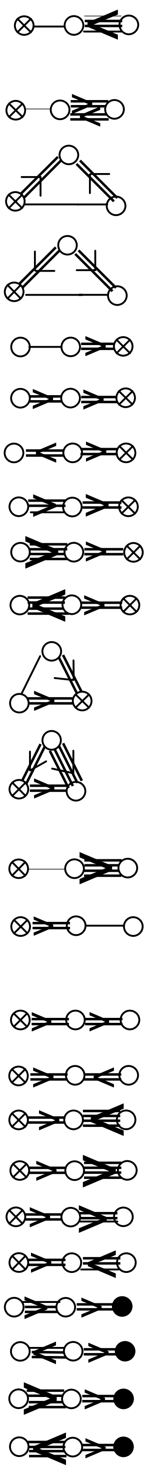
$$\text{for} \quad (\alpha, \alpha) \neq 0 \qquad \sigma_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha \tag{2.5}$$

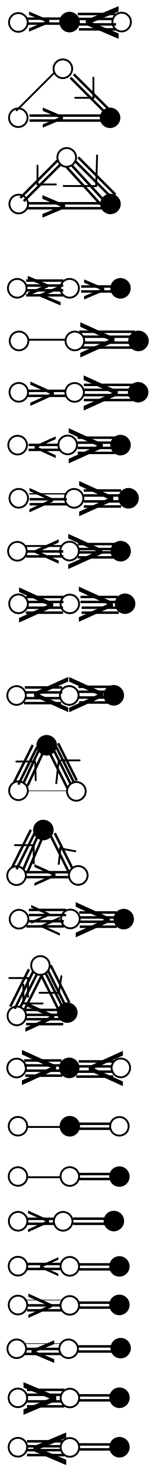
$$\text{for} \quad (\alpha, \alpha) = 0 \qquad \sigma_\alpha(\beta) = \begin{cases} \beta + \alpha & \text{if} \quad (\alpha, \beta) \neq 0 \\ \beta & \text{if} \quad (\alpha, \beta) = 0 \end{cases}$$

$$\sigma_\alpha(\alpha) = -\alpha. \tag{2.6}$$

The transformations (2.4) and (2.5) can be lifted to an automorphism of the algebra but transformations (2.6) cannot be lifted to an automorphism because even (odd) roots are transformed by σ_α into odd (even) ones and grading is not respected. However, transformations such as (2.6) are simply used to deduce from one simple root system all the other inequivalent root systems. The method is as follows. We construct from any α with $(\alpha, \alpha) = 0$ the system σ_α and then repeat the same operation on the obtained system until no new basis arises. It is an easy job (at least for lower rank) to construct from a given Dynkin diagram all other ones and it is noticed that only the roots linked to grey root, with respect to which the root system is

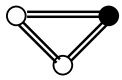
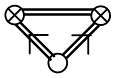
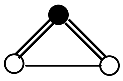
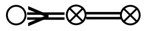
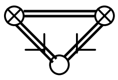
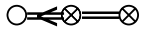
Table 1. Hyperbolic Kac–Moody superalgebras of rank 3.



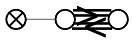


Dynkin diagrams	Subalgebras	Dynkin diagrams	Subalgebras
	$A_2^{(2)}, sl(1,1) \oplus A_1, A(0,1)$		$B^{(1)}(0,1), A_1 \oplus A_1, B(0,2)$
	$A_1^{(1)}, sl(1,1) \oplus A_1, A(0,1)$		$B(0,2), A_2$
	$B_2, B(1,1), A(0,1)$		$B^{(1)}(0,1), B_2, B(0,2)$
	$B_2, C(2), A(0,1)$		$B(0,2), A_1 \oplus B(0,1), A_1^{(1)}$
	$C(2), A_1 \oplus sl(1,1), A_2$		$B^{(1)}(0,1), A_1 \oplus B(0,1), A_2$
	$C(2), A_1 \oplus sl(1,1), B_2$		$B^{(1)}(0,1), A_1 \oplus B(0,1), B_2$
	$C(2), A_1 \oplus sl(1,1), B_2$		$B^{(1)}(0,1), A_1 \oplus B(0,1), B_2$
	$C(2), A_1 \oplus sl(1,1), G_2$		$B^{(1)}(0,1), A_1 \oplus B(0,1), G_2$
	$C(2), A_1 \oplus sl(1,1), A_2^{(2)}$		$B^{(1)}(0,1), A_1 \oplus B(0,1), G_2$
	$C(2), A_1 \oplus sl(1,1), A_2^{(2)}$		$B^{(1)}(0,1), A_1 \oplus B(0,1), A_2^{(2)}$
	$C(2), A_2$		$B^{(1)}(0,1), A_1 \oplus B(0,1), A_2^{(2)}$
	$A_2^{(2)}, C(2), B(1,1)$		$B^{(1)}(0,1), A_2$
	$A_2^{(2)}, sl(1,1) \oplus A_1, A(0,1)$		$B(0,2), B^{(1)}(0,1), B_2$
	$A_2, sl(1,1) \oplus A_1, B(1,1)$		$B^{(1)}(0,1), A_1 \oplus B(0,1), A_1^{(1)}$
	$B_2,\ sl(1,1) \oplus A_1, B_{(1,1)}$		$B^{(1)}(0,1), A_1^{(1)}$
	$B_2, sl(1,1) \oplus A_1, B(1,1)$		$B^{(1)}(0,1), A_1 \oplus A_1$
	$A_2^{(2)}, sl(1,1) \oplus A_1, B(1,1)$		$sl^{(4)}(1,3), A_1 \oplus A_1$
	$A_2^{(2)}, sl(1,1) \oplus A_1, B(1,1)$		$sl^{(4)}(1,3), A_1 \oplus B(0,1), A_2$
	$G_2, sl(1,1) \oplus A_1, B(1,1)$		$sl^{(4)}(1,3), A_1 \oplus B(0,1), B_2$
	$G_2, sl(1,1) \oplus A_1,\ B(1,1)$		$sl^{(4)}(1,3), A_1 \oplus B(0,1), B_2$
	$B(0,2), A_1 \oplus B(0,1), G_2$		$sl^{(4)}(1,3), A_1 \oplus B(0,1), G_2$
	$B(0,2), A_1 \oplus (0,1), G_2$		$sl^{(4)}(1,3), A_1 \oplus B(0,1), G_2$
	$B(0,2), A_1 \oplus B(0,1), A_2^{(2)}$		$sl^{(4)}(1,3), A_1 \oplus B(0,1), A_2^{(2)}$
	$B(0,2), A_1 \oplus B(0,1), A_2^{(2)}$		$sl^{(4)}(1,3), A_1 \oplus B(0,1), A_2^{(2)}$

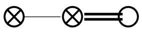
Table 1. (Continued.)

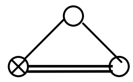
Dynkin diagrams	Subalgebras	Dynkin diagrams	Subalgebras
	$sl^{(4)}(1,3)$, $A_1^{(1)}$		$C(2)$
	$sl^{(4)}(1,3)$, A_2		$C(2)$, $A_1 \oplus sl(1,1)$
	$B(1,1)$, $C(2)$		$C(2)$, $A_1 \oplus sl(1,1)$, $B(1,1)$

transformed, will be affected. For example, the hyperbolic Kac–Moody superalgebra which is represented by the Dynkin diagram

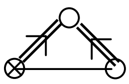


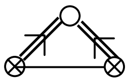
can also be represented by other Dynkin diagrams with inequivalent bases (distinguished and non-distinguished), i.e.





Similarly the following diagrams show how the same hyperbolic Kac–Moody superalgebra is represented by two Dynkin diagrams one with distinguished basis and the other with non-distinguished basis, i.e.





Following the above procedure now we can construct all the Dynkin diagrams of a given hyperbolic Kac–Moody superalgebra in a distinguished as well as non-distinguished root system.

3. Dynkin diagrams of hyperbolic Kac–Moody superalgebras

Following the procedure given in the previous section we first construct the Dynkin diagrams of hyperbolic Kac–Moody algebras of rank 2. Here we see that, similar to rank 2 hyperbolic Kac–Moody algebras, the number of Dynkin diagrams of superalgebra is also infinite, and deleting one of the vertices they reduce to $sl(2)$ ○ or $sl(1,1)$ ⊗ or $osp(1,2)$ ●. Hence all hyperbolic superalgebras of rank 2 are strictly hyperbolic and these are given by the Cartan matrix

$$\begin{pmatrix} 0 & k \\ -k' & 2 \end{pmatrix}$$

Table 2. Hyperbolic Kac–Moody superalgebras of rank 4.

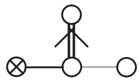
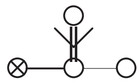
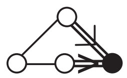
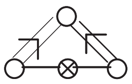
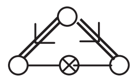
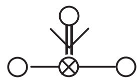
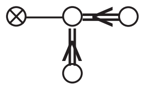
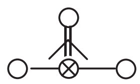
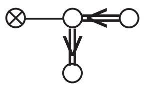
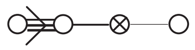
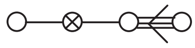
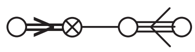
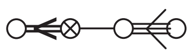
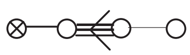
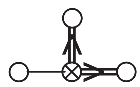
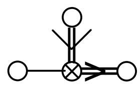
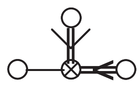
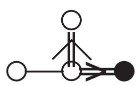
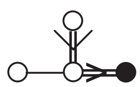
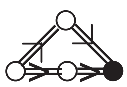
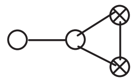
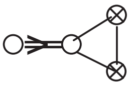
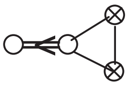
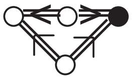
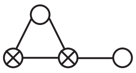
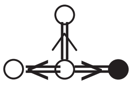
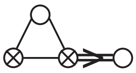
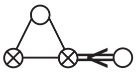
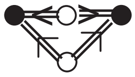
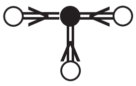
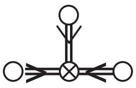
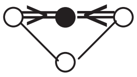
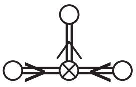
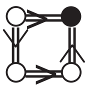
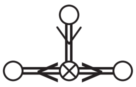
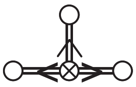
Dynkin diagrams	Subalgebras	Dynkin diagrams	Subalgebras
	B_3, $sl(1,1) \oplus A_1 \oplus A_1$, $A(1,1)$, $A(0,2)$		$A^{(2)}(0,3)$, $A_1 \oplus B(0,2)$, $A_2 \oplus A_1$, $B(0,3)$
	C_3, $sl(1,1) \oplus A_1 \oplus A_1$, $C(3)$, $A(0,2)$		$A^{(2)}(0,3)$, $B(0,3)$, A_3
	$B(2,1)$, $C_2^{(1)}$, $A(1,1)$		$D_3^{(2)}$, $sl(1,1) \oplus B_2$, $A(0,1) \oplus A_1$, $C(3)$
	$C(3)$, $D_3^{(2)}$, $A(1,1)$		$A_4^{(2)}$, $sl(1,1) \oplus B_2$, $A(0,1) \oplus A_1$, $C(3)$
	$D(2,1)$, $A_1 \oplus$ $A_1 \oplus A_1$, $A(1,1)$		$C_2^{(1)}$, $sl(1,1) \oplus A_1 \oplus A_1$, $C(3)$
	$B(1,2)$, $A_1 \oplus A_1 \oplus$ A_1, $A(1,1)$		$A_4^{(2)}$, $sl(1,1) \oplus A_1 \oplus A_1$ $B(2,1)$, $C(3)$
	$A(1,1)$, $A_1 \oplus A(0,1)$, $G_2 \oplus A_1$, $G(3)$		$G(3)$, $A_1 \oplus G_2$, $A(0,1) \oplus A_1$, $A(1,1)$
	C_3, $sl(1,1) \oplus A_2$, $A(0,1) \oplus A_1$, $B(2,1)$		$G(3)$, $A_1 \oplus G_2$, $C(2) \oplus A_1$, $D(2,1)$
	$A_4^{(2)}$, $sl(1,1) \oplus B_2$, $A(0,1) \oplus A_1$, $B(2,1)$		$G(3)$, $A_1 \oplus G_2$, $B(0,1) \oplus A_1$, $B(2,1)$
	$C_2^{(1)}$, $sl(1,1) \oplus B_2$, $A(0,1) \oplus A_1$, $B(2,1)$		$G_2^{(1)}$, $sl(1,1) \oplus A_2$, $A(0,1) \oplus A_1$, $G(3)$
	$A^{(2)}(2,2)$, $A_1 \oplus A_1 \oplus A_1$, $B(1,2)$		$B^{(1)}(0,2)$, $A_1 \oplus B(0,2)$, $A_2 \oplus B(0,1)$, B_3
	$B^{(1)}(1,1)$, $A_1 \oplus A_1 \oplus A_1$, $D(2,1)$, $B(1,2)$		$B^{(1)}(0,2)$, $A_1 \oplus B(0,2)$, $B_2 \oplus B(0,1)$, $A_4^{(2)}$
	$A^{(2)}(1,1)$, $A_1 \oplus$ $A_1 \oplus A_1$, $D(1,2)$		$B^{(1)}(0,2)$, $A_1 \oplus B(0,2)$, $B_2 \oplus B(0,1)$, $D_3^{(2)}$
	$A^{(4)}(0,4)$, $A_1 \oplus A_1 \oplus$ $B(0,1)$, B_3, $B(0,3)$		$A^{(2)}(0,3)$, $A_1 \oplus B(0,2)$, $B_2 \oplus A_1$, $B^{(1)}(0,2)$
	$B^{(1)}(0,2)$, $A_1 \oplus A_1 \oplus$ $B(0,1)$, C_3, $B(0,3)$		$A^{(2)}(0,3)$, $B^{(1)}(0,2)$, $D_3^{(2)}$

Table 2. (Continued.)

Dynkin diagrams	Subalgebras	Dynkin diagrams	Subalgebras
	$A^{(1)}(0,1)$, $A_1 \oplus A(0,1)$, $A(0,2)$		$A^{(4)}(0,4)$, $A_1 \oplus B(0,2)$, $B_2 \oplus B(0,1)$, $C_2^{(1)}$
	$A^{(1)}(0,1)$, $A_1 \oplus A(0,1)$, $C(3)$		$A^{(4)}(0,4)$, $A_1 \oplus B(0,2)$, $B_2 \oplus B(0,1)$, $A_4^{(2)}$
	$A^{(1)}(0,1)$, $A_1 \oplus A(0,1)$, $B(2,1)$		$A^{(2)}(0,3)$, $A^{(4)}(0,4)$, $C_2^{(1)}$
	$A(0,2)$, $A(0,1) \oplus A_1$, $A^{(1)}(0,1)$, $A(1,1)$		$D^{(2)}(2,1)$, $A_1 \oplus A_1 \oplus B(0,1)$, $A^{(2)}(2,2)$
	$B(2,1)$, $A(0,1) \oplus A_1$, $A^{(1)}(0,1)$, $B(1,2)$		$A^{(2)}(2,1)$, $A_1 \oplus A_1 \oplus B(0,1)$, $B^{(1)}(1,1)$, $D^{(2)}(2,1)$
	$C(3)$, $A(0,1) \oplus A_1$, $A^{(1)}(0,1)$, $D(2,1)$		$C^{(2)}(3)$, $A_1 \oplus B(0,2)$, $B(0,2) \oplus B(0,1)$, $A^{(2)}(0,3)$
	$A^{(2)}(0,3)$, $A_1 \oplus B(0,2)$, $B_2 \oplus A_1 A^{(4)}(0,4)$		$A^{(2)}(0,3)$, $C^{(2)}(3)$, $A^{(2)}(0,3)$
	$A^{(2)}(0,3)$, $A_1 \oplus A_1 \oplus A_1$		$A^{(2)}(1,1)$, $A_1 \oplus A_1 \oplus A_1$
	$B(0,3)$, A_3, $A^{(2)}(0,3)$		$B^{(1)}(1,1)$, $A_1 \oplus A_1 \oplus A_1$, $A^{(2)}(1,1)$
	$A^{(4)}(0,4)$, $C_3^{(1)}$, $A^{(2)}(0,3)$		$B^{(1)}(1,1)$, $A_1 \oplus A_1 \oplus A_1$, $A^{(2)}(2,2)$
	$A^{(4)}(0,4)$, $A_1 \oplus B(0,2)$, $A_2 \oplus B(0,1)$, C_3		$A^{(2)}(2,2)$, $A_1 \oplus A_1 \oplus A_1$

where kk' is greater than 4. The corresponding Dynkin diagram is given by

where the thick line is equal to kk' number of lines and the arrow can point in either or both directions. Similarly, for rank 2 hyperbolic superalgebra another type of Cartan matrix is

Table 3. Hyperbolic Kac–Moody superalgebras of rank 5.

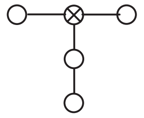
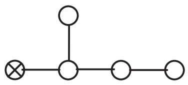
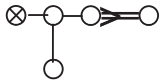
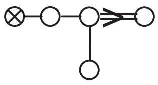
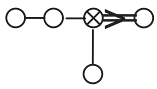
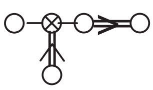
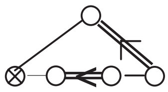
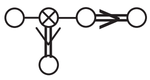
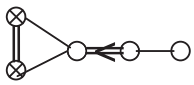
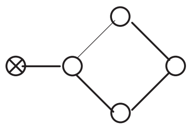
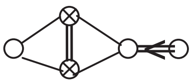
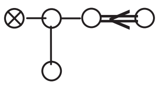
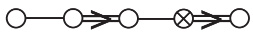
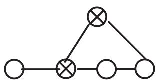
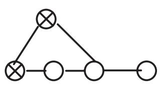
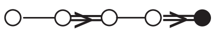
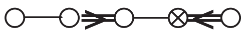
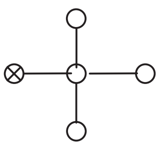
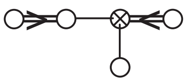
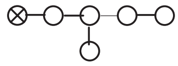
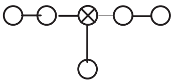
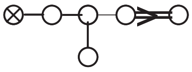
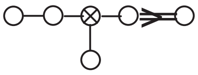
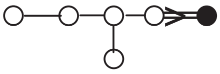
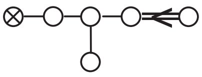
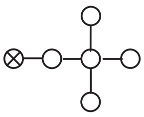
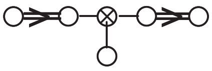
Dynkin diagrams	Subalgegbras	Dynkin diagrams	Subalgebras
	$A(2,1)$, $A_1 \oplus A_1 \oplus A_2$, $D(2,1,\alpha) \oplus$ A_1, $D^{(1)}(2,1,\alpha)$		$F(4)$, $A_1 \oplus B_3$, $A(0,1) \oplus$ A_2, $A(1,1) \oplus A_1$, $D(2,2)$
	A_4, $sl(1,1) \oplus A_1 \oplus A_2$, $A(0,2) \oplus A_1$, $D(3,1)$, $A(0,3)$		$F(4)$, $A_1 \oplus B_3$, $C(2) \oplus A_2$, $D(2,1) \oplus A_1$, $A^{(2)}(1,3)$
	F_4, $sl(1,1) \oplus C_3$, $A(0,1) \oplus$ A_2, $A(0,2) \oplus A_1$, $B(3,1)$		$F(4)$, $A_1 \oplus B_3$, $B(1,1) \oplus$ A_2, $B(1,2) \oplus A_1$, $B^{(1)}(1,2)$
	B_4, $sl(1,1) \oplus A_1 \oplus B_2$, $A(0,2) \oplus A_1$, $D(3,1)$, $B(3,1)$		B_4, $sl(1,1) \oplus A_3$, $A(0,1) \oplus A_2$, $C(3) \oplus A_1$, $F(4)$
	$B_3^{(1)}$, $sl(1,1) \oplus B_3$, $A(0,1) \oplus A_1 \oplus A_1$, $A(0,3)$, $B(3,1)$		$A_6^{(2)}$, $sl(1,1) \oplus C_3 A(0,1) \oplus$ B_2, $C(3) \oplus A_1$, $F(4)$
	$A^{(2)}(2,3)$, $A_1 \oplus B(1,2)$, $A_2 \oplus A_1 \oplus A_1$, $A(2,1)$, $B(1,3)$		$D_4^{(2)}$, $sl(1,1) \oplus B_3$, $A(0,1) \oplus B_2$, $C(3) \oplus A_1$, $F(4)$
	$B^{(1)}(2,1)$, $A_1 \oplus A_1 B_2$, $D(2,1) \oplus A_1$, $D^{(1)}(2,1)$, $B(2,2)$		$D_4^{(2)}$, , $F(4)$, $D(2,2)$
	$A^{(2)}(2,4)$, $A_1 \oplus A_1 \oplus B_2$, $B(1,2) \oplus A_1$, $A^{(2)}(2,3)$, $B(2,2)$		$F(4)$, $C(2) \oplus A_2$, $C(3) \oplus A_1$, $C^{(1)}(3)$
	$A_4^{(1)}$, $sl(1,1) \oplus A_3$, $A(0,3)$, $D(2,1)$		$C^{(1)}(3)$, $D(2,2)$, $C(2) \oplus A_1$, $A^{(2)}(1,3)$
	C_4, $sl(1,1) \oplus A_1 \oplus B_2$, $A(0,2) \oplus A_1$, $D(2,1)$, $C(4)$		$B^{(1)}(1,2)$, $A_1 \oplus B(1,2)$, $A_2 \oplus B(1,1)$, $B_3 \oplus A_1$, $F(4)$
	$A(1,2)$, $D(3,1)$, $A(0,3)$, $A(0,2) \oplus A_1$, $A^{(1)}(0,2)$		$A^{(2)}(4,1)$, $A_1 \oplus B(2,1)$, $A_2 \oplus B(1,1)$, $B_3 \oplus B(0,1)$, F_4
	$D(3,1)$, $A(0,2)$, $A(0,2) \oplus$ A_1, $A^{(2)}(0,2)$		$B^{(1)}(0,3)$, $A_1 \oplus B(0,3)$, $A_2 \oplus B(0,2)$, $B_3 \oplus$ $B(0,1)$, F_4
	$B^{(1)}(1,2)$, $A_1 \oplus B(1,2)$, $A_2 \oplus B(1,1)$, $F(4)$		$A^{(2)}(1,3)$, $A_1 \oplus D(2,1)$, $A_2 \oplus C(2)$, $B_3 \oplus A_1$, $F(4)$
	D_4, $sl(1,1) \oplus A_1 \oplus$ $A_1 \oplus A_1$, $D(2,1)$		$D^{(1)}(2,1)$, $A_1 \oplus D(2,1)$, $B_2 \oplus A_1 \oplus A_1$, $D(2,2)$, $A^{(2)}(1,3)$

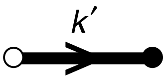
Table 4. Hyperbolic Kac–Moody superalgebras of rank 6.

Dynkin diagrams	Subalgebras
	D_5, $sl(1,1) \oplus A_4$, $A(0,1) \oplus A_2 \oplus A_1$, $A(0,3) \oplus A_1$, $D(4,1)$, $A(0,4)$
	$D(2,3)$, $A_1 \oplus A(1,2)$, $A_2 \oplus A_1 \oplus A_1$, $D(2,3)$, $A(2,2)$
	$B_4^{(1)}$, $sl(1,1) \oplus B_4$, $A(0,1) \oplus A_1 \oplus B_2$, $D(4,1)$, $B(1,4)$
	$F_4^{(1)}$, $sl(1,1) \oplus F_4$, $A(0,1) \oplus C_3$, $A(0,2) \oplus A_2$, $A(0,3) \oplus A_1$, $B(1,4)$
	$A^{(2)}(2,5)$, $A_1 \oplus B(2,2)$, $A_2 \oplus B_2 \oplus A_1$, $A(1,2) \oplus A_1$, $D(2,3)$
	$A^{(2)}(0,7)$, $A_1 \oplus B(0,4)$, $A_2 \oplus A_1 \oplus B(0,2)$, $A_4 \oplus B(0,1)$, D_5, $B(0,5)$
	$A_7^{(2)}$, $sl(1,1) \oplus C_4$, $A(0,1) \oplus B_2 \oplus A_1$, $A(0,3) \oplus A_1$, $D(4,1)$, $C(5)$
	$E_6^{(2)}$, $sl(1,1) \oplus F_4$, $A(0,1) \oplus B_3$, $A(0,2) \oplus A_2$, $A(0,3) \oplus A_1$, $C(5)$
	$D_4^{(1)}$, $sl(1,1) \oplus D_4$, $A(0,1) \oplus A_1 \oplus A_1 \oplus A_1$, $D(4,1)$
	$A^{(2)}(2,5)$, $A_1 \oplus B(2,2)$, $B_2 \oplus A_1 \oplus B_2$, $D(2,2) \oplus A_1$, $D^{(1)}(2,2)$, $B^{(1)}(2,2)$

given by

$$\begin{pmatrix} 2 & -1 \\ -k' & 2 \end{pmatrix}$$

where k' is an even integer greater than 4 with its Dynkin diagram given by



.

Similarly, the Dynkin diagrams of hyperbolic Kac–Moody superalgebras of ranks 3, 4, 5 and 6 are displayed in tables 1, 2, 3 and 4, respectively, in the distinguished basis. Interestingly, we observe that there is no hyperbolic Kac–Moody superalgebra with rank 7 or more. In the case of hyperbolic Kac–Moody algebra, the classification halts at rank 10; this is essentially because the maximum rank of the finite and affine exceptional series stops at E_8 (rank 8) and $E_8{}^{(1)}$ (rank 9) respectively. Thus, higher rank hyperbolic Kac–Moody algebra is not possible as there is no affine exceptional algebra beyond $E_8{}^{(1)}$ to support it. Similarly, the maximum rank of the exceptional Lie superalgebra is 4 and exceptional finite and affine series stop at rank 4 and 5, receptively. Now we pass from a finite Lie superalgebra to an affine one by adding a single new light direction to root space and then from affine to hyperbolic with a second light-like root independent of the first. As a result the maximum allowed rank of hyperbolic Kac–Moody superalgebra halts at 6 (running parallel with the non-super case). This is indeed the case when we check using a case-by-case approach. In the case of hyperbolic Kac–Moody algebra, we see that there is no strictly hyperbolic case with rank more than 4, but in the case of hyperbolic superalgebra there is no such restriction.

4. Conclusion

In conclusion we would like to add a few remarks concerning the Dynkin diagrams of hyperbolic Kac–Moody superalgebras so obtained. These diagrams themselves can be exploited to serve various purposes. For example, they can be used to determine the maximal regular subalgebras by deleting arbitrarily one of the vertices. From the Dynkin diagrams of an algebra, we can easily draw the Satake super diagrams [9] of the corresponding algebras, which can be used to determine the real forms and associated symmetric super spaces. Such studies are currently in progress. As a continuation of this paper, next we have shown how the involutive automorphisms obtained from the Satake super diagrams are used to furnish a general treatment of Iwasawa decomposition [10–12] of these algebras with specific examples.

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